Newman-Janis method and rotating dilaton-axion black hole

S. Yazadjiev *

Department of Theoretical Physics, Faculty of Physics, Sofia University,

5 James Bourchier Boulevard, Sofia 1164, Bulgaria

February 7, 2008

Abstract

It's shown that the rotating dilaton-axion black hole solution can be obtained from GGHS static charged dilaton black hole solution via Newman-Janis method.
PACS number(s): 02.30.Dk,04.20.Cv

The low energy limit of the heterotic string theory gives an interesting generalization of Einstein-Maxwell theory - the Einstein-Maxwell-dilaton - axion gravity.The field equations of the Einstein-Maxwell-dilaton-axion gravity in four dimensions can be obtained from the following action [1],[2]

$$\mathcal{A} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}e^{4\varphi}\partial_\mu\Theta\partial^\mu\Theta + e^{-2\varphi}F_{\mu\mu}F^{\mu\nu} + \Theta F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \quad (1)$$

Here R is the Ricci scalar with respect to the space-time metric $g_{\mu\nu}$(with a signature $(+,-,-,-)$), φ is the dilaton field, $F_{\mu\nu} = (dA)_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are correspondingly the Maxwell tensor and its dual, the pseudo scalar Θ is related to the Kalb-Ramond field $H^{\mu\nu\sigma}$ through the relation

$$H^{\mu\nu\sigma} = \frac{1}{2}e^{4\varphi}\varepsilon^{\mu\nu\sigma\rho}\partial_\rho\Theta.$$

In last decade the string black holes attract much attention.The static spherically symmetric charged dilaton black hole was obtained by Gibbons [3] and independently by Garfinkle,Horowitz and Strominger [4].Using the string target space duality rotation Sen found the rotating dilaton-axion black hole solution generating it from Kerr solution [5].

It's well-known that Kerr and Kerr-Newman solution in Einstein theory can be generated correspondingly from Schwarzschild and Reissner-Nordsrom solution

*E-mail: yazad@phys.uni-sofia.bg

via Newman-Janis method [6],[7]. It's natural to ask whether Sen's rotating dilaton-axion solution can be obtained via Newman-Janis method from GGHS dilaton black hole solution.

The purpose of the present note is to show that the rotating dilaton-axion black hole solution can be "derived" from static spherically symmetric dilaton black hole solution via Newman-Janis procedure.

Here we will not discuss the Newman-Janis algorithm in details.We refer the reader to the recent papers [8],[9].It should be noted, however that in Newman-Janis procedure there is a certain arbitrariness and an element of guess.

The GGHS dilaton black hole solution may be written in different coordinates and there is no pure physical reasons which of them are more appropriate for our purpose. It seems to be natural to expect that the desirable coordinates in which the GGHS solution should be written are these obtained by generating the GGHS solution directly from Schwarzschild solution. The generating the GGHS solution from Schwarzschild's one has been already done in [10].Here we give the final result

$$ds^2 = \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right) dt^2 - \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right)^{-1} dr^2 - r^2\left(1+\frac{r_2}{r}\right)\left(d\theta^2 + \sin^2(\theta) d\phi^2\right) \qquad (2)$$

$$e^{2\varphi} = \frac{1}{1+\frac{r_2}{r}}$$

$$\Phi = -\frac{\frac{Q}{r}}{1+\frac{r_2}{r}}$$

where φ is the dilaton and Φ is the electric potential. The parameters r_1 and r_2 are given by

$$r_1 + r_2 = 2\mathcal{M}$$

and

$$r_2 = \frac{Q^2}{\mathcal{M}}$$

where $\mathcal{M}$ and Q are the mass and the charge of the dilaton black hole.

Following Newman and Janis (see also [8] and [9]) the first step is to write the metric (2) in advanced Eddington-Finkelstein coordinates.Performing the coordinate transformation

$$dt = du + \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right)^{-1} dr \qquad (3)$$

we obtain

$$ds^2 = \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right) du^2 + 2du dr - r^2\left(1+\frac{r_2}{r}\right) d\Omega^2 \qquad (4)$$

This metric may be presented in terms of its null tetrad vectors

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \qquad (5)$$

where

$$
\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right)\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}r\sqrt{1+\frac{r_2}{r}}}\left(\delta_2^\mu + \frac{i}{\sin(\theta)}\delta_3^\mu\right)
\end{aligned}
\tag{6}
$$

Let's now the radial coordinate r allowed to take the complex values, as keeping the null vectors l^μ and n^μ real and $\bar{m}^\mu$ complex conjugated to m^μ. Then the tetrad takes the form

$$
\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left(\frac{1-\frac{r_1}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)}{1+\frac{r_2}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)}\right)\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}\bar{r}\sqrt{1+\frac{r_2}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)}}\left(\delta_2^\mu + \frac{i}{\sin(\theta)}\delta_3^\mu\right)
\end{aligned}
\tag{7}
$$

The next step is to perform formally the complex coordinate transformation

$$
\begin{aligned}
r' &= r + ia\cos(\theta) \qquad \theta' = \theta \\
u' &= u - ia\cos\theta \qquad \phi' = \phi
\end{aligned}
\tag{8}
$$

By keeping r' and u' real we obtain the following tetrad

$$
\begin{aligned}
l'^\mu &= \delta_1^\mu \\
n'^\mu &= \delta_0^\mu - \frac{1}{2}\left(\frac{1-\frac{r_1 r'}{\Sigma}}{1+\frac{r_2 r'}{\Sigma}}\right)\delta_1^\mu \\
m'^\mu &= \frac{1}{\sqrt{2}(r'+ia\cos(\theta))}\frac{1}{\sqrt{1+\frac{r_2 r'}{\Sigma}}}\left(ia\cos(\theta)(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin(\theta)}\delta_3^\mu\right)
\end{aligned}
\tag{9}
$$

where $\Sigma = r'^2 + a^2\cos^2(\theta)$.

The metric formed by this tetrad is (dropping the primes)

$$
g^{\mu\nu} = \begin{pmatrix}
-\frac{a^2\sin^2(\theta)}{\tilde{\Sigma}} & 1+\frac{a^2\sin^2(\theta)}{\tilde{\Sigma}} & 0 & -\frac{a}{\tilde{\Sigma}} \\
* & -e^{2U(r,\theta)} - \frac{a^2\sin^2(\theta)}{\tilde{\Sigma}} & 0 & \frac{a}{\tilde{\Sigma}} \\
* & * & -\frac{1}{\tilde{\Sigma}} & 0 \\
* & * & * & -\frac{1}{\tilde{\Sigma}}
\end{pmatrix}
\tag{10}
$$

where we have put

$$
e^{2U(r,\theta)} = \left(\frac{1-\frac{r_1 r}{\Sigma}}{1+\frac{r_2 r}{\Sigma}}\right)
\tag{11}
$$

and

$$\tilde{\Sigma} = \left(1 + \frac{r_2 r}{\Sigma}\right)\Sigma = r(r + r_2) + a^2\cos^2(\theta) \tag{12}$$

The corresponding covariant metric is

$$g_{\mu\nu} = \begin{pmatrix} e^{2U(r,\theta)} & 1 & 0 & a\sin^2(\theta)\left(1 - e^{2U(r,\theta)}\right) \\ * & 0 & 0 & -a\sin^2(\theta) \\ * & * & -\tilde{\Sigma} & 0 \\ * & * & * & -sin^2(\theta)\left(\tilde{\Sigma} + a^2\sin^2(\theta)\left(2 - e^{2U(r,\theta)}\right)\right) \end{pmatrix} \tag{13}$$

A further simplification is made by the following coordinate transformation

$$du = dt' - \frac{\Delta_2}{\Delta}dr \qquad d\phi = d\phi' - \frac{a}{\Delta}dr \tag{14}$$

where $\Delta = r(r - r_1) + a^2$ and $\Delta_2 = r(r + r_2) + a^2$. This transformation leaves only one off-diagonal element and the metric takes the form (dropping the primes on t and ϕ)

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = e^{2U(r,\theta)}dt^2 - \frac{\tilde{\Sigma}}{e^{2U(r,\theta)}\tilde{\Sigma} + a^2\sin^2(\theta)}dr^2 - \tilde{\Sigma}d\theta^2 + \\ 2a\sin^2(\theta)\left(1 - e^{2U(r,\theta)}\right)dtd\phi - \sin^2(\theta)\left(\tilde{\Sigma} + a^2\sin^2(\theta)\left(2 - e^{2U(r,\theta)}\right)\right)d\phi^2 \end{aligned} \tag{15}$$

Taking into account that $r_1 + r_2 = 2\mathcal{M}$ we obtain

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2\mathcal{M}r}{\tilde{\Sigma}}\right)dt^2 - \tilde{\Sigma}\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \\ \frac{4\mathcal{M}ra\sin^2(\theta)}{\tilde{\Sigma}}dtd\phi - \left(r(r + r_2) + a^2 + \frac{2\mathcal{M}ra^2\sin^2(\theta)}{\tilde{\Sigma}}\right)\sin^2(\theta)d\phi^2 \end{aligned} \tag{16}$$

where $e^{2U(r,\theta)}\tilde{\Sigma} + a^2\sin^2(\theta) = r(r - r_1) + a^2 = \Delta$.

This is the rotating dilaton-axion black hole metric [5].The other quantities are given by

$$\begin{aligned} A &= -\frac{Qr}{\tilde{\Sigma}}\left(dt - a\sin^2(\theta)d\phi\right) \\ e^{2\varphi} &= \frac{1}{1 + \frac{r_2 r}{\Sigma}} = \frac{\Sigma}{\tilde{\Sigma}} = \frac{r^2 + a^2\cos^2(\theta)}{r(r + \frac{Q^2}{\mathcal{M}}) + a^2\cos^2(\theta)} \\ \Theta &= \frac{Q^2}{\mathcal{M}}\frac{a\cos(\theta)}{\Sigma} = \frac{Q^2}{\mathcal{M}}\frac{a\cos(\theta)}{r^2 + a^2\cos^2(\theta)} \end{aligned} \tag{17}$$

It's useful to present the metric (16) in the form

$$ds^2 = e^{2U}(dt + \omega_i dx^i) - e^{-2U}h_{ij}dx^i dx^j \tag{18}$$

After a few algebra we find

$$ds^2 = e^{2U(r,\theta)}\left(dt + \frac{2\mathcal{M}ar\sin^2(\theta)}{\tilde{\Sigma}_1}d\phi\right)^2 - \tag{19}$$
$$e^{-2U(r,\theta)}\left(\tilde{\Sigma}_1\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \Delta\sin^2(\theta)d\phi^2\right)$$

where

$$\tilde{\Sigma}_1 = \tilde{\Sigma} - 2\mathcal{M} = r(r-r_1) + a^2\cos^2(\theta) \tag{20}$$
$$e^{2U(r,\theta)} = 1 - \frac{2\mathcal{M}r}{\tilde{\Sigma}} = \frac{1}{1-\frac{2\mathcal{M}r}{\tilde{\Sigma}_1}}.$$

It should be expected that using the Newamn-Janis method we will able to generate stationary axisymmetric solutions starting with static spherically symmetric solutions different from the GGHS solution.For example, using as seed solutions the three classes two-parametric families of solutions presented in [10], it should be expected that we will obtain the corresponding rotating naked singularities in Einstein-Maxwell-dilaton-axion gravity.

There are some questions which arise.As we have seen the Newman-Janis method generates the rotating solution of Einstein-Maxwell-dilaton-axion starting with GGHS solution in proper coordinates.GGHS solution, however, is also a solution to the truncated theory without axion field (i.e. Einstein-Maxwell-dilaton gravity).Why the Newman-Janis method does not generate the rotating solution to truncated model instead to the full model? In our opinion the cause is probably that the full theory in the presence of two commuting Killing's vectors possesses larger nontrivial symmetry group than the truncated model.

Acknowledgments

The author wishes to express his thanks to P.Fiziev for his continuous encouragement and the stimulating conversations.

This work was partially supported by the Sofia University Foundation for Scientific Researches, Contract No. 245/99, and by the Bulgarian National Foundation for Scientific Researches, Contract F610/99.

References

[1] M. Green, J. Schwarz, E. Witten, Superstring theory, Cambridge University Press, Cambridge, England (1987)

[2] J. Scherk, J. Schwarz, Nucl.Phys. **B 153** 61 (1973)

[3] G. Gibbons, Nucl.Phys. **B 207**, 337 (1982)

[4] D. Garfinkle, G. Horowitz, A. Strominger, Phys.Rev.**D43**, 3140 (1991); 45, 3888(E) (1992)

[5] A. Sen, Phys.Rev.Lett. **69** , 1006 (1992)

[6] E. Newman, A. Janis, J.Math.Phys. **6**, 915 (1965)

[7] E. Newman,E. Couch,K. Chinnapared,A. Exton,A Prakash, R. Torrence, J.Math.Phys. **6**, 918 (1965)

[8] S. Drake, R. Turolla, Class.Quant.Grav. **14**, 1883 (1997)

[9] S. Drake, P. Szekeres, **An explanation of the Newman-Janis algorithm**, E-print: gr-qc/9807001

[10] S. Yazadjiev, **Exact static solutions in four dimensional Einstein-Maxwell-Dilaton gravity**, E-print: gr-qc/9906048